MATH 2060 TUT03

Thm 6.4.1 (Taylor's Thm)

Let
- $n \in \mathbb{N}$
- $f : [a,b] \longrightarrow \mathbb{R}$ s.t. $(a < b)$
- $f', \dots, f^{(n)}$ are cts on $[a,b]$ and
- $f^{(n+1)}$ exists on (a,b).

If $x_0 \in [a,b]$, then $\forall x \in [a,b]$, $\exists\, c$ between x_0 and x s.t.

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{P_n(x)} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}}_{R_n(x)}$$

$P_n(x)$: n-th Taylor's Polynomial of f at x_0

$R_n(x)$: remainder (Lagrange form)

3. Use Induction to prove Leibniz's rule for the nth derivative of a product:

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x).$$

Ans: Let $P(n)$ be the statement $(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x)\, g^{(k)}(x)$

- $P(1)$ is just product rule. So $P(1)$ is true.
- Suppose $P(m)$ is true for some $m \in \mathbb{N}$.

i.e. $(fg)^{(m)} = \sum_{k=0}^{m} \binom{m}{k} f^{(m-k)} g^{(k)}$ $(*)$

Diff. both sides of $(*)$:

$$\begin{aligned}
(fg)^{m+1}(x) &= \sum_{k=0}^{m} \binom{m}{k} \frac{d}{dx}\left(f^{(m-k)} g^{(k)}\right) \\
&= \sum_{k=0}^{m} \binom{m}{k} \left(f^{(m+1-k)} g^{(k)} + f^{(m-k)} g^{(k+1)}\right) \\
&= f^{(m+1)} g^{(0)} + \sum_{k=1}^{m} \binom{m}{k} f^{(m+1-k)} g^{(k)} \\
&\qquad + \sum_{k=0}^{m-1} \binom{m}{k} f^{(m-k)} g^{(k+1)} + f^{(0)} g^{(m+1)} \\
&= f^{(m+1)} g^{(0)} + \sum_{k=1}^{m} \binom{m}{k} f^{(m+1-k)} g^{(k)} \\
&\qquad + \sum_{k=1}^{m} \binom{m}{k-1} f^{(m+1-k)} g^{(k)} + f^{(0)} g^{(m+1)} \\
&= f^{(m+1)} g^{(0)} + \sum_{k=1}^{m} \binom{m+1}{k} f^{(m+1-k)} g^{(k)} + f^{(0)} g^{(m+1)} \\
&= \sum_{k=0}^{m+1} \binom{m+1}{k} f^{(m+1-k)} g^{(k)} \qquad \left(\binom{m+1}{0} = \binom{m+1}{m+1} = 1\right)
\end{aligned}$$

So $P(m+1)$ is true.

- By Mathematical Induction, $P(n)$ is true $\forall n \in \mathbb{N}$. //

7. If $x > 0$ show that $|(1+x)^{1/3} - (1 + \frac{1}{3}x - \frac{1}{9}x^2)| \le (5/81)x^3$. Use this inequality to approximate $\sqrt[3]{1.2}$ and $\sqrt[3]{2}$.

Ans: Let $f(x) = (1+x)^{1/3}$, which is indefinitely diff. $\forall x > 0$.

Try to apply Taylor's Thm.

Take $x_0 = 0$, $n = 2$.

Note $f'(x) = \frac{1}{3}(1+x)^{-2/3}$ $\Rightarrow$ $f'(0) = \frac{1}{3}$

$f''(x) = -\frac{2}{9}(1+x)^{-5/3}$ $\quad$ $f''(0) = -\frac{2}{9}$

$f'''(x) = \frac{10}{27}(1+x)^{-8/3}$

So $P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + \frac{1}{3}x - \frac{1}{9}x^2$

Let $x > 0$. By Taylor's Thm,

$$f(x) = P_2(x) + R_2(x)$$

where $R_2(x) = \frac{1}{3!}f'''(c)x^3 = \frac{5}{81}(1+c)^{-8/3}x^3$

for some $c \in (0, x)$

Hence $|f(x) - P_2(x)| = \frac{5}{81}(1+c)^{-8/3}x^3$

$\le \frac{5}{81}x^3$ $\quad$ $(c > 0 \Rightarrow (1+c)^{-8/3} \le 1)$

- $\sqrt[3]{1.2} = f(0.2) \approx P_2(0.2) = 1 + \frac{2}{30} - \frac{4}{900} = \frac{239}{225} \approx 1.0622$

  with error $\le \frac{5}{81}(0.2)^3 = \frac{1}{2025}$

- $\sqrt[3]{2} = f(1) \approx P_2(1) = 1 + \frac{1}{3} - \frac{1}{9} = \frac{11}{9} \approx 1.2222$

  with error $\le \frac{5}{81}(1)^3 = \frac{5}{81}$

//

8. If $f(x) := e^x$, show that the remainder term in Taylor's Theorem converges to zero as $n \to \infty$, for each fixed x_0 and x. [*Hint:* See Theorem 3.2.11.]

Ans: Note f is indefinitely differentiable on $\mathbb{R}$ and
$f^{(n)}(x) = e^x \quad \forall n \in \mathbb{N},\ \forall x \in \mathbb{R}$.

Fix $x_0, x \in \mathbb{R}$. Assume $x_0 \neq x$.
The n-th remainder term in Taylor's Thm is

$$R_n(x) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-x_0)^{n+1} \text{ for some } c_n \text{ between } x_0, x.$$

$$= \frac{e^{c_n}}{(n+1)!}(x-x_0)^{n+1}$$

$$\Rightarrow |R_n(x)| \leq \frac{e^M |x-x_0|^{n+1}}{(n+1)!} \quad \text{where } M \geq |x_0|, |x|.$$

$$=: a_n$$

Want: $\lim(a_n) = 0$. Use Ratio Test!

Note $\lim\limits_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim\limits_{n\to\infty}\left(\frac{|x-x_0|}{n+1}\right)$

$$= 0 < 1$$

By Ratio Test, $\lim\limits_{n\to\infty}(a_n) = 0$

Therefore $\lim\limits_{n\to\infty}(R_n(x)) = 0$ by Squeeze Thm. //

10. Let $h(x) := e^{-1/x^2}$ for $x \neq 0$ and $h(0) := 0$. Show that $h^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Conclude that the remainder term in Taylor's Theorem for $x_0 = 0$ does not converge to zero as $n \to \infty$ for $x \neq 0$.

Ans: Clearly $h(x)$ is infinitely diff. for $x \neq 0$

Apply Leibniz's rule to $h'(x) = \frac{2}{x^3} e^{-1/x^2} = \frac{2}{x^3} h(x)$, $x \neq 0$:

$$h^{(n+1)}(x) = \frac{d^n}{dx^n}\left(\frac{2}{x^3} h(x)\right) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{2}{x^3}\right)^{(n-k)} h^{(k)}(x)$$

$$= \sum_{k=0}^{n} \binom{n}{k} (2)(-3)(-4)\cdots(-(n-k+2))\, x^{-(n-k+3)}\, h^{(k)}(x)$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (n-k+2)!\, \frac{h^{(k)}(x)}{x^{n-k+3}} \qquad (*)$$

We prove by induction on $n \geq 0$ that

$$\left.\begin{aligned} &1) \quad \lim_{x\to 0} h^{(n)}(x)/x^m = 0 \quad \forall\, m \in \mathbb{N} \\ &2) \quad h^{(n+1)}(0) = 0. \end{aligned}\right\} P(n)$$

When $n = 0$: 1) $\displaystyle \lim_{x\to 0} \frac{h(x)}{x^m} = \lim_{x \to 0} \frac{(1/x^2)^m}{e^{1/x^2}} \cdot x^m = 0$

since $\displaystyle e^{1/x^2} \geq \frac{(1/x^2)^m}{m!} \Rightarrow \frac{(1/x^2)^m}{e^{1/x^2}} \leq m!$

2) $\displaystyle h'(0) = \lim_{x\to 0} \frac{h(x) - h(0)}{x - 0} = 0$.

So $P(0)$ is true.

Suppose $P(k)$ is true for $k=0,1,\dots,n$

Now :

1) By (*),

$$\lim_{x\to 0} \frac{h^{(n+1)}(x)}{x^m} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (n-k+2)! \overbrace{\left(\lim_{x\to 0} \frac{h^{(k)}(x)}{x^{n-k+3+m}}\right)}^{=0 \text{ by induction assumption.}} = 0$$

2) $$h^{(n+2)}(0) = \lim_{x\to 0} \frac{h^{(n+1)}(x) - \cancelto{0}{h^{(n+1)}(0)}}{x-0} = \lim_{x\to 0} \frac{h^{(n+1)}(x)}{x} = 0$$

This completes the induction.

Finally, $$R_n(x) = h(x) - \sum_{k=0}^{n} \frac{\cancelto{0}{h^{(k)}(0)}}{k!} x^k = h(x)$$

and so $\lim_{n\to\infty} R_n(x) = h(x) \neq 0$ for $x \neq 0$ //

So h is indefinitely diff. on $\mathbb{R}$ but its Taylor series at 0 does not converge to h itself (except when $x=0$).

Example Consider a function f whose second derivative $f''(x)$ exists and is continuous on $[0,1]$. Assume that $f(0)=f(1)=0$ and suppose that there exists $A>0$ such that $|f''(x)| \le A$ for $x \in [0,1]$. Show that

$$\left|f'\left(\frac{1}{2}\right)\right| \le \frac{A}{4} \quad \text{and} \quad |f'(x)| \le \frac{A}{2}$$

Ans: Apply Taylor's Thm to f with center $1/2$, we have $\forall x \in [0,1]$, $\exists\, c_x$ between $1/2$, x s.t

$$f(x) = f(\tfrac{1}{2}) + f'(\tfrac{1}{2})(x-\tfrac{1}{2}) + \frac{f''(c_x)}{2!}(x-\tfrac{1}{2})^2$$

Put $x=1$:

$$0 = f(1) = f(\tfrac{1}{2}) + f'(\tfrac{1}{2})(\tfrac{1}{2}) + f''(c_1)(\tfrac{1}{2^3}) \quad ①$$

Put $x=0$:

$$0 = f(0) = f(\tfrac{1}{2}) + f'(\tfrac{1}{2})(-\tfrac{1}{2}) + f''(c_0)(\tfrac{1}{2^3}) \quad ②$$

① - ② :

$$0 = f'(\tfrac{1}{2}) + \tfrac{1}{8}\left(f''(c_1) - f''(c_0)\right)$$

$$\Rightarrow \quad |f'(\tfrac{1}{2})| \le \tfrac{1}{8}\left(|f''(c_1)| + |f''(c_0)|\right) \le \tfrac{1}{8}(A+A) = \frac{A}{4}.$$

Apply Taylor's Thm to f with center a :

- If $0 < a \le 1$, $\exists\, c_0 \in (0,a)$ s.t.

$$0 = f(0) = f(a) + f'(a)(0-a) + \frac{f''(c_0)}{2!}(0-a)^2 \quad ③$$

- If $0 \le a < 1$, $\exists\, c_1 \in (a,1)$ s.t.

$$0 = f(1) = f(a) + f'(a)(1-a) + \frac{f''(c_1)}{2!}(1-a)^2 \quad ④$$

Now, $\forall a \in (0,1)$

④-③ :

$$0 = f'(a) + \tfrac{1}{2}\left[f''(c_1)(1-a)^2 - f''(c_0)a^2\right]$$

$$\Rightarrow \quad |f'(a)| \le \frac{A}{2}\left[(1-a)^2 + a^2\right] \le \frac{A}{2}\left[(1-a) + a\right] = \frac{A}{2}$$

If $a=0$, ④ $\Rightarrow$ $|f'(0)| = \tfrac{1}{2}|f''(c_1)| \le \frac{A}{2}$

If $a=1$, ③ $\Rightarrow$ $|f'(1)| = \tfrac{1}{2}|f''(c_0)| \le \frac{A}{2}$ //